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Geometrical Interpretation of Spinors[1]

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Geometrical properties of elements of the unique representation of the Clifford algebra of quadratic form $(+, -, -, -)$ are investigated. A connection between horospheres on the positive Lobatschevsky space of timelike directions and spinors is established.

1. INTRODUCTION

The main purpose of this paper is to investigate a geometrical interpretation of spinors in special relativity where spinors are defined as some elements of the Clifford algebra generated by vectors of Minkowski space [the vector space equipped with quadratic form of signature $(+, -, -, -)$]. We begin with an investigation of the connection between spinors and Minkowski space, and give a construction of an isomorphism χ of the Hermitian part of the tensor product of the so-called odd and even half-spinors into Minkowski space. We have to restrict the symmetry group of the spinor space to the so-called $\mathrm{Spin}_+ \cong SL(2, C)$ group in order to assure the existence of invariant skew bilinear forms on the spaces of two-spinors. We obtain known (Penrose, 1968; Trautman, 1965) results concerning relations of vectors to Hermitean matrices and so on, except in the case when the two-spinor space Σ is not an abstract two-dimensional complex vector space with a skew bilinear form, but is generated by vectors of Minkowski space by the Clifford product. We obtain in a simple manner the well-known Penrose world flag and show that this flag is invariant with respect to the Crumeyrolle group $\mathscr{H}$, which is the group that determines the existence of a spinor structure on a space-time manifold E. Finally, we establish a connection between horospheres on the positive Lobatschevsky

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space of timelike directions (that is, orbits of the group $\mathscr{H}$) taken with parametrizations on them, and spinors of the space Σ. This connection is interesting with respect to a physical interpretation of the reduction of the Lorentz principal bundle of orthonormal frames to the Crumeyrolle group.

2. SPINORS AND MINKOWSKI SPACE

Given a quadratic form Q on an even-dimensional vector space M, by the spinor space we understand the vector space of the unique, up to equivalence, irreducible representation of the Clifford algebra $C(Q)$ of the quadratic form Q. In the case of Minkowski space, the form Q is given by diag $(+, -, -, -)$. Let (e_0, e_1, e_2, e_3) be the orthonormal base of E. Then the Clifford algebra C has (Bourbaki, 1959) the underlying vector space isomorphic to the vector space of the exterior algebra $\wedge E$ of E, which means that C is spanned by the products $e_{i_1} \cdots e_{i_k}$, with $0 \leqslant i_1, i_2, \ldots, i_k \leqslant 3$, and by unit. Thus the space E will be considered as a subspace of C. The multiplication law in C will be completely defined by the quadratic form in such a way that for any $x, y \in E$,

$$xy = x \wedge y + \tfrac{1}{2}B(x, y) \tag{2.1}$$

with $B(x, y) = Q(x + y) - Q(x) - Q(y) = 2x \cdot y$. "$\cdot$" denotes the scalar product, "$\wedge$" denotes the exterior product. Hence we have

$$x^2 = Q(x) \cdot 1 \tag{2.2}$$

We see that the exterior algebra $\wedge E$ may be identified with the Clifford algebra of the zero form on E.

To obtain the spinor space S we should pass to the complexifications E_C, Q', C' of E, Q, C, respectively. Now every orthogonal frame $\epsilon = (e_0, e_1, e_2, e_3)$ gives rise to the Witt base $\omega = (x_1, x_2, y_1, y_2)$, where

$$\begin{aligned} x_1 &= \tfrac{1}{2}(e_0 + e_3) \qquad & y_1 &= \tfrac{1}{2}(e_0 - e_3) \\ x_2 &= \tfrac{1}{2}(ie_1 + e_2) \qquad & y_2 &= \tfrac{1}{2}(ie_1 - e_2) \end{aligned} \tag{2.3}$$

We see that x_1 and x_2 span the totally singular subspace N as well as y_1 and y_2 span the totally singular subspace P of E_C, which are mutually supplementary and

$$\begin{aligned} B'(x_i, x_j) &= B'(y_i, y_j) = 0 \\ B'(x_i, y_j) &= \delta_{ij} \end{aligned} \tag{2.4}$$

(here B' is the extension of the bilinear form B to E_C). Let us set $f = y_1 y_2$. Now $C'f$ will be the minimal left ideal of C', and the Clifford algebra C' is isomorphic to the algebra of endomorphisms of the vector space $C'f$

spanned by $(y_1y_2, x_1y_1y_2, x_2y_1y_2, x_1x_2y_1y_2)$. Therefore we shall identify $C'f$ with the spinor space S of the quadratic form $Q(1, 3)$. If we denote by C^N and C^P the subalgebras of C' generated by N and P, respectively ($C^N \cong \wedge N$, $C^P \cong \wedge P$), then we have $C'f \cong C^N f$. The representation ρ' of C' on C^N is given by

$$(\rho'(v) \circ u)f \stackrel{\text{def}}{=} vuf \stackrel{\text{def}}{=} \rho(v) \circ uf \tag{2.5}$$

for every $v \in C'$, $u \in C^N$. Hence we may take the space S of spinors to be $S = C'f = C^N$, and ρ (as well as ρ') as the spin representation. As a base of this spinor space we can take

$$\mathscr{S} = (y_1y_2, x_1y_1y_2, x_2y_1y_2, x_1x_2y_1y_2) \equiv (1, x_1, x_2, x_1x_2) \tag{2.6}$$

We can distinguish two subspaces of S: S_0 spanned by x_1 and x_2, and S_e spanned by 1 and x_1x_2, which will be called the spaces of odd and even half-spinors, respectively.

The spinor representation ρ (ρ') of C' gives rise to spinor representations of some groups contained in the Clifford algebra C'. Namely, some elements s of C' form a multiplication group of invertible elements of C' with property $sxs^{-1} \in E$ for any $x \in E$, called the Clifford group G. From (2.2) we have $Q(sxs^{-1})\cdot 1 = (sxs^{-1})^2 = Q(x)\cdot 1$, so we obtain the map $\varphi: G \to O(1, 3)$. The group G has subgroups Pin, Spin, and Spin_+, which are the covering groups of $\mathscr{L} = O(1, 3)$, $SO(1, 3)$, and $SO_+(1, 3) = \mathscr{L}_0$, respectively. It appears, however, that the spinor representation ρ (ρ') of $\text{Spin}_+ \cong SL(2, C)$ is a sum of two inequivalent, irreducible representations on the half-spinor spaces S_0 and S_e.

Now for future use we introduce the notion of the main antiautomorphism α of C'. It can be defined by a listing of its properties:

$$\begin{aligned} \alpha^2 &= I \\ \alpha(u\cdot v) &= \alpha(v)\alpha(u) && \text{for every } u, v \in C' \\ \alpha(x) &= x && \text{for every } x \in E \\ \alpha(a\cdot 1) &= a\cdot 1 && \text{for every } a \in \mathbb{C} \\ \alpha(xy) &= yx = B(x, y)\cdot 1 - xy && \text{for every } x, y \in E \end{aligned} \tag{2.7}$$

So for $f \in C'$ we have

$$\alpha(f) = -f \tag{2.8}$$

Now we have a possibility of introducing a bilinear form β on the spinor space S as given by

$$\alpha(uf)vf = \beta(u, v)f \qquad \text{for every } u, v \in C^N \tag{2.9}$$

or equivalently,

$$\alpha(u)v = \beta(u, v)x_1x_2 \tag{2.10}$$

It is easy to see (Chevalley, 1954; Crumeyrolle, 1969) that the form is nondegenerate and skew symmetric. Besides, it is the zero form on $S_0 \times S_e$ and $S_e \times S_0$, and

$$\beta(\rho'(s)u, \rho'(s)v) = \beta(u, v) \qquad \text{for } s \in \text{Spin}_+,\ u, v \in C^N \tag{2.11}$$

In this manner we reach two two-dimensional vector spaces S_0 and S_e with the Spin_+ as a symmetry group, equipped with bilinear skew form β invariant with respect to the group Spin_+.

Let us consider now the space $S \otimes S$, i.e., the tensor product of the spinor space S with itself. It is the space of representation $\rho' \otimes \rho'$ of Spin_+, defined by

$$\rho' \otimes \rho'(s)(u \otimes v) = \rho'(s)u \otimes \rho'(s)v \tag{2.12}$$

The mapping ψ from $S \otimes S$ into the Clifford algebra C' given by

$$\psi(u \otimes v) = uf\alpha(v) \qquad \text{for every } u, v \in S \cong C^N \tag{2.13}$$

appears to be a linear isomorphism, for we show that $\psi(S \otimes S)$ is a two-sided ideal different from 0 of the simple algebra C', and hence it is isomorphic to C'. For every $w \in C'$ we have

$$wuf\alpha(v) = (\rho'(w)u)f\alpha(v) \in \psi(S \otimes S) \tag{2.14}$$

and

$$uf\alpha(v)w = u\alpha(\alpha(w)v\alpha(f)) = -u\alpha(\alpha(w)vf) = uf\alpha(\rho'(\alpha(w))v) \in \psi(S \otimes S) \tag{2.15}$$

Moreover, for $s \in \text{Spin}_+$ we have

$$\begin{aligned}\psi(\rho'(s)u \otimes \rho'(s)v) &= suf\alpha(\rho'(s)v) = -su\alpha(\rho'(s)vf)\\ &= suf\alpha(v)\alpha(s)ss^{-1} = suf\alpha(v)s^{-1}\end{aligned} \tag{2.16}$$

because for every $s \in \text{Spin}_+$ its norm given by

$$N(s) \coloneqq \alpha(s)s \tag{2.17}$$

is equal to 1.

Now we shall find the relation between the tensor product of half-spinors and Minkowski space E. As it has been said earlier, we identify E with a subspace of the Clifford algebra C. So from (2.16) we have that if we transform spinor space by some element $s \in \text{Spin}_+$, then Minkowski space $E \in C$ is transformed by some element belonging to $SO_+(1, 3) = \mathscr{L}_0$. From (2.3) we obtain

$$\begin{aligned} &e_0 = x_1 + y_1 \qquad && e_2 = x_2 - y_2 \\ &e_1 = \frac{1}{i}(x_2 + y_2) \qquad && e_3 = x_1 - y_1 \\ &(e_0{}^2 = 1 = -e_1{}^2 = -e_2{}^2 = -e_3{}^2) && \end{aligned} \tag{2.18}$$

Let us denote the base elements of S_0 and S_e by (ρ, σ) and (ρ^*, σ^*), respectively. So we have

$$S_o = \{x_1, x_2\} = \{\rho, \sigma\} \tag{2.19}$$

with the form β

$$\beta(\rho, \sigma) = 1 = -\beta(\sigma, \rho) \tag{2.20}$$

and

$$S_e = \{1, x_1x_2\} = \{-\sigma^*, \rho^*) \tag{2.21}$$

with

$$\beta(\rho^*, \sigma^*) = -\beta(\sigma^*, \rho^*) = 1 \tag{2.22}$$

We introduce an isomorphism χ between $S_0 \otimes S_e$ and E_C given by

$$\chi = (1 + \alpha) \circ \psi \tag{2.23}$$

that is,

$$\chi(u \otimes v) = \psi(u \otimes v) - \psi(v \otimes u) \tag{2.24}$$

which preserves the transformation property (2.16):

$$\chi(\rho'(s)u \otimes \rho'(s)v) = s\chi(u \otimes v)s^{-1} \tag{2.25}$$

for every $s \in \text{Spin}_+$. It can be verified that

$$\begin{aligned} \chi(\rho \otimes \rho^*) &= x_1 \\ \chi(\sigma \otimes \rho^*) &= x_2 \\ \chi(\sigma \otimes \sigma^*) &= y_1 \\ \chi(\rho \otimes \sigma^*) &= -y_2 \end{aligned} \tag{2.26}$$

and

$$\chi(\rho^* \otimes \sigma^*) + \chi(\rho \otimes \sigma) = 1 \tag{2.27}$$

Now for any $u_i \in S_0$ and $v_i \in S_e$ we can calculate that

$$\chi(u_1 \otimes v_1)\chi(u_2 \otimes v_2) = \beta(u_1, u_2)\psi(v_1 \otimes v_2) + \beta(v_1, v_2)\psi(u_1 \otimes u_2) \tag{2.28}$$

where on the left-hand side we have the Clifford product of two vectors: $\chi(u_1 \otimes v_1)$ and $\chi(u_2 \otimes v_2)$.

Since from (2.1) we have the scalar product of two vectors

$$x \cdot y = \tfrac{1}{2}B'(x, y) = \tfrac{1}{2}(xy + yx) \tag{2.29}$$

we have, taking into account (2.13), (2.9), and (2.28),

$$B'(\chi(u_1 \otimes v_1), \chi(u_2 \otimes v_2)) = \beta(u_1, u_2)\beta(v_1, v_2) \tag{2.30}$$

for $u_i \in S_0$, $v_i \in S_e$. Thus every vector obtained from $u \otimes v$ by χ is a null one, although in the general case it is a complex one.

If we omit the symbol χ denoting the isomorphism of $S_0 \otimes S_e$ into E_C, we can write (2.18) and (2.26) in the known form (Penrose, 1968; Trautman, 1965):

$$\begin{aligned} e_0 &= \rho \otimes \rho^* + \sigma \otimes \sigma^* \\ e_1 &= \frac{1}{i}(\sigma \otimes \rho^* - \rho \otimes \sigma^*) \\ e_2 &= \sigma \otimes \rho^* + \rho \otimes \sigma^* \\ e_3 &= \rho \otimes \rho^* - \sigma \otimes \sigma^* \end{aligned} \tag{2.31}$$

In this manner every vector of the Minkowski space E can be described by the matrix with components:

$$\begin{pmatrix} \rho \otimes \rho^* & \rho \otimes \sigma^* \\ \sigma \otimes \rho^* & \sigma \otimes \sigma^* \end{pmatrix} \tag{2.32}$$

or equivalently, by (2.26) and (2.3),

$$\begin{pmatrix} x_1 & -y_2 \\ x_2 & y_1 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} e_0 + e_3 & e_2 - ie_1 \\ e_2 + ie_1 & e_0 - e_3 \end{pmatrix} \tag{2.33}$$

So any vector $x \in E$ can be represented by the matrix

$$x = x^i \sigma_i = x^0 \sigma_0 + \mathbf{x} \cdot \boldsymbol{\sigma} \tag{2.34}$$

where from (2.33)

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad \sigma_1 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \quad \sigma_2 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{2.35}$$

It has been mentioned already that the group Spin_+ acts in different ways on the spaces S_0 and S_e. Namely, it is easy to see that the Lie algebra of Spin_+ generated by $e_i e_j$ is represented in $S_e = \{\rho^*, \sigma^*\}$ by matrices, which are complex conjugate with respect to the matrices representing the same generators in $S_0 = \{\rho, \sigma\}$. Thus for any $s \in \text{Spin}_+$ we have

$$su = \wedge u \quad \text{and} \quad sv = \overline{\wedge} v, \quad \text{where } \wedge \in SL(2, C),\ u \in S_0,\ v \in S_e \tag{2.36}$$

This allows us to define the anti-isomorphism "$*$" of S_0 into S_e:

$$*: u = u^0 \rho + u^1 \sigma \to u^* = \bar{u}^0 \rho^* + \bar{u}^1 \sigma^* \tag{2.37}$$

with properties

$$(su)^* = su^* \tag{2.38}$$

and

$$\overline{\beta(u_1, u_2)} = \beta(u_1^*, u_2^*) \tag{2.39}$$

for every $s \in \text{Spin}_+ = SL(2, C)$.

We will denote the space S_0 by Σ and the space S_e by Σ^*. Now let us check the transformation law of the matrix x representing a vector x from E. Let x be given by $x = u \otimes v^* + v \otimes u^*$, where $u = \alpha\rho + \beta\sigma \equiv \binom{\alpha}{\beta}$ and $v = \gamma\rho^* + \delta\sigma^* \equiv \binom{\gamma}{\delta}$. Because

$$\begin{pmatrix} \alpha \\ \beta \end{pmatrix} \otimes \begin{pmatrix} \gamma \\ \delta \end{pmatrix} \xrightarrow{\wedge \in SL(2,C)} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha \\ \beta \end{pmatrix} \otimes \begin{pmatrix} \bar{a} & \bar{b} \\ \bar{c} & \bar{d} \end{pmatrix} \begin{pmatrix} \gamma \\ \delta \end{pmatrix} \tag{2.40}$$

we obtain for $\begin{pmatrix} \alpha\gamma & \alpha\delta \\ \beta\gamma & \beta\delta \end{pmatrix}$ that

$$\begin{pmatrix} \alpha\gamma & \alpha\delta \\ \beta\gamma & \beta\delta \end{pmatrix} \xrightarrow{\wedge} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \alpha\gamma & \alpha\delta \\ \beta\gamma & \beta\delta \end{pmatrix} \begin{pmatrix} \bar{a} & \bar{c} \\ \bar{b} & \bar{d} \end{pmatrix} \tag{2.41}$$

Thus when the spinor space is transformed by $\wedge \in SL(2, C)$ then E is transformed by the appropriate element of $\mathscr{L}_0$ according to

$$\hat{x} \rightsquigarrow \hat{x}' = \bigwedge \hat{x} \bar{\bigwedge}^T \tag{2.42}$$

3. PENROSE WORLD FLAGS

Let us take any spinor $u \in \Sigma$. The anti-isomorphism $*: \Sigma \to \Sigma^*$ gives us the spinor $u^* \in \Sigma^*$. Thus by the isomorphism $\chi: \Sigma \otimes \Sigma^* \to E$ we obtain

$$u \otimes u^* \rightsquigarrow \text{light vector of Minkowski space}$$

The group $SL(2, C)$ acts transitively on the space Σ [if we discard the point $\binom{0}{0}$]; hence we can limit our attention to the case $u = \rho$. Owing to the existence of the invariant skew bilinear form β on Σ, we can introduce a spinor $\tilde{\sigma} \in \Sigma$ which has the property

$$\beta(\rho, \tilde{\sigma}) = 1 \tag{3.1}$$

It can be easily seen that $\tilde{\sigma}$ is defined by (3.1) up to $\alpha\rho$, with $\alpha \in \mathbb{C}$, i.e.,

$$\tilde{\sigma} = \sigma + \alpha\rho \tag{3.2}$$

The spinor ρ gives us under χ the null vector

$$x_1 = \rho \otimes \rho^* \tag{3.3}$$

and vectors

$$k = \rho \otimes \tilde{\sigma}^* + \tilde{\sigma} \otimes \rho^* = \rho \otimes \sigma^* + \sigma \otimes \rho^* + 2 \operatorname{Re} [\alpha(\rho \otimes \rho^*)] \tag{3.4}$$

defined up to ax_1 with $a \in \mathbb{R}$. From (2.29), (2.30), (2.20), and (2.22) we find that

$$k \cdot k = -1 \tag{3.5}$$

and

$$k \cdot x_1 = 0 \tag{3.6}$$

Thus it appears that the spinor ρ defines the null vector $x_1 = \rho \otimes \rho^*$ and some plane P, spanned by x_1 and k, and tangent to the light cone along the null direction defined by x_1. This is the known Penrose world flag connected with the spinor ρ (Penrose, 1968) [or the so-called isotropic straight line in direction of x_1 in negative Lobatschevsky space (Gelfand, 1962)]. Now let us take the positive Lobatschevsky space given in a fixed base frame (e_0, e_1, e_2, e_3) by the equation

$$x^0 = 1 \tag{3.7}$$

This hyperplane intersects the light cone "along" the sphere S^2 of radius 1. Let ξ be a light vector in the direction of x_1, and such that $\xi \in S^2$. There exists only one vector $k_0 \in P$ tangent to S^2 at ξ (k_0 fulfils the condition

$k_0 \cdot r = 0$ with the uniquely defined vector $r = \xi - e_0$). (In the language of isotropic straight lines it means that by the realization of the hyperspace $x^0 = 1$ the isotropic straight line in the direction of ξ passes into a straight line tangent to S^2.) We find that in our case

$$k_0 = \rho \otimes \sigma^* + \sigma \otimes \rho^* \tag{3.8}$$

Let us take another spinor $\rho' = e^{i\varphi}\rho$, obtained from ρ by the transformation

$$\rho' = \Lambda \rho \qquad \text{with} \qquad \Lambda = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix} \in SL(2, C) \tag{3.9}$$

From (2.36) we have

$$\rho'^* = \bar{\Lambda} \rho^* \qquad \text{so} \qquad \rho' \otimes \rho'^* = \rho \otimes \rho^* \tag{3.10}$$

According to (2.32) and (2.42) we obtain

$$\hat{k}_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \overset{\Lambda}{\rightsquigarrow} \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix} \begin{pmatrix} 0 & 1 \\ & 0 \end{pmatrix} \begin{pmatrix} e^{-i\varphi} & 0 \\ 0 & e^{i\varphi} \end{pmatrix} = \begin{pmatrix} 0 & e^{2i\varphi} \\ e^{-2i\varphi} & 0 \end{pmatrix} = \hat{k}_0' \tag{3.11}$$

Hence the spinor ρ' defines another world flag P' by ξ, which is rotated with respect to P by the angle 2φ.

Now let us take the subgroup $\mathscr{H}$ of $SL(2, C)$ (the so-called spinorality group) of the elements of the form

$$\varkappa \in \mathscr{H} \equiv \varkappa = \begin{pmatrix} 1 & \xi \\ 0 & 1 \end{pmatrix} \qquad \text{with} \qquad \varkappa \in \mathbb{C} \tag{3.12}$$

Again in agreement with (2.42), we see that

$$\begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} = \xi \overset{\varkappa}{\rightsquigarrow} \varkappa \xi \bar{\varkappa}^T = \xi \tag{3.13}$$

and

$$k_0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \rightsquigarrow \begin{pmatrix} \varkappa + \bar{\varkappa} & 1 \\ 1 & 0 \end{pmatrix} = k_0 + (2 \operatorname{Re} \varkappa)\xi \tag{3.14}$$

so the world flag defined by the spinor ρ is invariant under any transformation of the spinorality group $\mathscr{H}$. (This is to be expected because the spinor remains unmoved by any transformation belonging to $\mathscr{H}$, as can be easily checked.)

4. SPINORS AND HOROSPHERES

In this section we shall try to see spinors from another point of view. Accordingly, let us take the Lobatschevsky space Y of timelike directions of

Minkowski space E. Let us realize this space as the set of points of the upper part of the hyperbola:

$$x \cdot x = \tfrac{1}{2}B(x, x) = 1 \tag{4.1}$$

The distance r between two different points $x, y \in Y$ is given by

$$\cosh r = x \cdot y \geqslant 1 \tag{4.2}$$

and equals zero only when the points coincide. Let us notice that in this metric space the distance between x and y tends to infinity, when the time direction defined by $y \in Y$ tends to a null ray. Thus the null rays appear as points in infinity of our Lobatschevsky space. As from (4.2) we see that the distance between two points is unchanged by the Lorentz transformation, the Lorentz orthochronous group is the group of motions of the space Y, and the space Y is the homogeneous space of $\mathscr{L}_0$.

Now we shall introduce a horosphere ω on Y as an orbit of some point $x \in Y$ with respect to the spinorality group $\mathscr{H}$, or the group conjugate to it:

$$\omega_0 = x\mathscr{H} \tag{4.3}$$

or

$$\omega = x\mathscr{H}g = xgg^{-1}\mathscr{H}g = yg\mathscr{H}g^{-1} \tag{4.4}$$

with $g \in \mathscr{L}_0$ (Gelfand, 1962).

In such a way the horosphere ω is given by some point $x \in Y$ and some element $g \in \mathscr{L}_0$. So we see that the set of horospheres is transitive under the group $\mathscr{L}_0$, and from (2.34) and (2.42) we see that it is the homogeneous space of the group $SL(2, C)$. We know from Section 3 that the spinor ρ defines uniquely the null vector $\xi \in S^2$ as well as the spacelike vector k_0 tangent to S^2 at $\xi(k_0{}^2 = -1)$. Now let us choose a hyperplane containing the origin of coordinates, and intersecting the light cone. It is known that every such hyperplane is defined by the equation

$$x \cdot t = 0 \tag{4.5}$$

where t is some spacelike vector, $t \cdot t < 0$. Thus choosing the hyperplane $\mathscr{H}$ perpendicular to k_0 we see that $\xi \in \mathscr{H}$. Now it is possible to select another null vector $\xi' \in S^2$ which has the properties

$$\xi' \in \mathscr{H} \qquad \text{that is} \qquad \xi' \cdot k_0 = 0 \tag{4.6}$$

and

$$\xi' \cdot \xi = 2 \tag{4.7}$$

Thus every spinor ρ allows us to select unique timelike vector x:

$$x = k_0 + \xi + \frac{1}{\xi \cdot \xi'}\,\xi' \tag{4.8}$$

which has properties

$$x \cdot \xi = 1 \tag{4.9}$$

and

$$x \cdot x = 1 \tag{4.10}$$

In the considered case of the spinor ρ and basis (ρ, σ) of Σ we have

$$\xi' = 2\sigma \otimes \sigma^* \tag{4.11}$$

and

$$x = \rho \otimes \sigma^* + \sigma \otimes \rho^* + 2\rho \otimes \rho^* + \sigma \otimes \sigma^* \tag{4.12}$$

so from (2.36) we have

$$x \rightsquigarrow \hat{x} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \tag{4.13}$$

Because x belongs to the space Y, we see that the spinor ρ defines the horosphere

$$\omega = x\mathscr{H} = \left\{ \begin{pmatrix} 2 + \varkappa + \bar{\varkappa} + \varkappa\bar{\varkappa} & 1 + \varkappa \\ 1 + \bar{\varkappa} & 1 \end{pmatrix} \right\}_{\varkappa \in \mathbb{C}} \tag{4.14}$$

Now from (2.42) we obtain that the vector x' related to the spinor $\rho' = e^{i\varphi}\rho$ will be equal to

$$x' = \begin{pmatrix} 2 & e^{2i\varphi} \\ e^{-2i\varphi} & 1 \end{pmatrix} \tag{4.15}$$

so x' belongs to ω and is obtained from x by the transformation belonging to $\mathscr{H}$ and defined by $\varkappa = e^{2i\varphi} - 1$. This should be expected, because the group $\mathscr{H}$ is invariant with respect to the elements of the form

$$\begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}$$

that is,

$$\begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix} \begin{pmatrix} 1 & \varkappa \\ 0 & 1 \end{pmatrix} \begin{pmatrix} e^{-i\varphi} & 0 \\ 0 & e^{i\varphi} \end{pmatrix} = \begin{pmatrix} 1 & e^{2i\varphi} \\ 0 & 1 \end{pmatrix} \in \mathscr{H} \tag{4.16}$$

In such a way we have obtained that the space of horospheres of the positive Lobatschevsky space Y is a homogeneous space of the group $SL(2, C)$, with the group

$$\begin{pmatrix} e^{i\varphi} & \varkappa \\ 0 & e^{-i\varphi} \end{pmatrix}$$

as a stabilizer group. But it is known that this group is the stabilizer group of the homogeneous space of light rays. But all homogeneous spaces with conjugate stabilizers can be identified, so every horosphere ω is defined uniquely by a null vector $\xi \in S^2$. Although the whole class of spinors, defined by the subgroup of elements

$$\begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix}$$

of the group $SL(2, C)$, gives the same horosphere, we can say that two different spinors of this class define different coordinate systems on the two-dimensional Euclidean space of this horosphere. Namely, let ρ define x by (4.8). Then every element of the horosphere

$$\omega = x\mathscr{H} = x \circ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix}$$

given by ρ, can be parametrized by z, i.e., by two real numbers a and b such that $z = a + ib$. Now for $\rho' = e^{i\varphi}\rho$ we have

$$\omega = \omega' = x' \wedge^{-1} \mathscr{H} \wedge = x'\mathscr{H}_\wedge = x \wedge \mathscr{H}_\wedge = x\begin{pmatrix} e^{i\varphi} & e^{i\varphi}z \\ 0 & e^{-i\varphi} \end{pmatrix}$$

$$\wedge = \begin{pmatrix} e^{i\varphi} & 0 \\ 0 & e^{-i\varphi} \end{pmatrix} \qquad (4.17)$$

It seems attractive to admit that the spinor $\rho' = e^{i\varphi}\rho$ defines a parametrization of our horosphere $\omega = \omega'$ by $e^{i\varphi}z$, i.e., by two real numbers a' and b' such that

$$\begin{aligned} a' &= a\cos\varphi - b\sin\varphi \\ b' &= a\sin\varphi + b\cos\varphi \end{aligned} \qquad (4.18)$$

Although such a parametrization would be very useful as it allows us to distinguish between two spinors $e^{i\varphi}\rho$ and $e^{i(\varphi+\pi)}\rho$ in terms of the geometry of Minkowski space, nevertheless it seems desirable to define a parametrization of the horosphere ω which agrees with the action of the group $\mathscr{H}$ on ω. More exactly, let us take once again the horosphere ω defined by ρ. Then for every $x = (x^0, x^1, x^2, x^3) \in \omega$ we have

$$x \rightsquigarrow x' = x \circ \begin{pmatrix} 1 & z \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} x^0 + x^3 + a^2 + b^2 + 2(ax^2 - bx^1) & x^2 + a - i(x^1 - b) \\ (x^2 + a) + i(x^1 - b) & x^0 - x^3 \end{pmatrix} \qquad (4.19)$$

since $x^0 - x^3 = 1$ for all $x \in \omega$. We see that the group $\mathscr{H}$ acts on elements $x' \in \omega \subset Y$ by translation $(-b, a)$ of coordinates (x^1, x^2), i.e.,

$$\begin{aligned} x^{1\prime} &= x^1 - b \\ x^{2\prime} &= x^2 + a \end{aligned} \qquad (4.20)$$

while

$$\begin{aligned} x^3 \rightsquigarrow x^{3\prime} &= x^3 + \frac{a^2 + b^2}{2} + ax^2 - bx^1 \\ x^0 \rightsquigarrow x^{0\prime} &= x^0 + \frac{a^2 + b^2}{2} + ax^2 - bx^1 \end{aligned} \qquad (4.21)$$

On the other hand, from (4.17) we obtain

$$x \rightsquigarrow x' = x \circ \begin{pmatrix} e^{i\varphi} & e^{i\varphi}\varkappa \\ 0 & e^{-i\varphi} \end{pmatrix} \tag{4.22}$$

with

$$\begin{aligned} x^{0\prime} &= x^0 + \frac{a^2 + b^2}{2} + ax^2 - bx^1 \\ x^{1\prime} &= (x^1 - b)\cos 2\varphi - (x^2 + a)\sin 2\varphi \\ x^{2\prime} &= (x^1 - b)\sin 2\varphi + (x^2 + a)\cos 2\varphi \\ x^{3\prime} &= x^3 + \frac{a^2 + b^2}{2} + ax^2 - bx^1 \end{aligned} \tag{4.23}$$

Thus comparing (4.20) and (4.23), we see that the spinor $e^{i\varphi}\rho$ involves a change of the reference frame in (x^1, x^2) given by the rotation on 2φ. Therefore the parametrizations of ω, defined by the action of $\mathscr{H}$ [(4.20), (4.23)] will be identical for spinors $e^{i\varphi}\rho$ and $e^{i(\varphi+\pi)}\rho$. This result is in agreement with the fact that the vector k_0 [and thus x from (4.8)] given by ρ coincides with the vector k_0 given by $e^{i\pi}\rho$. In terms of the world flags it means that spinors $e^{i\varphi}\rho$ and $e^{i(\varphi+\pi)}\rho$ define the same world flag.

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